DAY FIFTEEN

Indefinite Integrals

Learning & Revision for the Day

- Integral as an Anti-derivative
- Fundamental Integration Formulae
- Methods of Integration

Integral as an Anti-derivative

A function $\phi(x)$ is called a **primitive** or **anti-derivative** of a function $f(x)$, if $\phi'(x) = f(x)$. If $f_1(x)$ and $f_2(x)$ are two anti-derivatives of $f(x)$, then $f_1(x)$ and $f_2(x)$ differ by a constant. The collection of all its anti-derivatives is called **indefinite integral** of $f(x)$ and is denoted by $\int f(x)\,dx$.

Thus, $\dfrac{d}{dx}\{\phi(x) + C\} = f(x) \Rightarrow \int f(x)\,dx = \phi(x) + C$

where, $\phi(x)$ is an anti-derivative of $f(x)$, $f(x)$ is the **integrand** and C is an arbitrary constant known as the **constant of integration**. Anti-derivative of odd function is always even and of even function is always odd.

Properties of Indefinite Integrals

- $\int \{f(x) \pm g(x)\}\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
- $\int k \cdot f(x)\,dx = k \cdot \int f(x)\,dx$, where k is any non-zero real number.
- $\int [k_1 f_1(x) + k_2 f_2(x) + \ldots + k_n f_n(x)]\,dx = k_1 \int f_1(x)\,dx + k_2 \int f_2(x)\,dx + \ldots + k_n \int f_n(x)\,dx$, where $k_1, k_2, \ldots k_n$ are non-zero real numbers.

Fundamental Integration Formulae

There are some important fundamental formulae, which are given below

1. Algebraic Formulae

(i) $\int x^n dx = \dfrac{x^{n+1}}{n+1} + C,\ n \neq -1$

(ii) $\int (ax+b)^n dx = \dfrac{1}{a} \cdot \dfrac{(ax+b)^{n+1}}{n+1} + C,\ n \neq -1$

(iii) $\int \frac{1}{x} dx = \log|x| + C$

(iv) $\int \frac{1}{ax+b} dx = \frac{1}{a}(\log|ax+b|) + C$

(v) $\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log\left|\frac{a+x}{a-x}\right| + C$

(vi) $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + C$

(vii) $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$

(viii) $\int \frac{-1}{a^2 + x^2} dx = \frac{1}{a} \cot^{-1}\left(\frac{x}{a}\right) + C$

(ix) $\int \frac{1}{\sqrt{x^2 - a^2}} dx = \log|x + \sqrt{x^2 - a^2}| + C$

(x) $\int \frac{1}{\sqrt{x^2 + a^2}} dx = \log|x + \sqrt{x^2 + a^2}| + C$

(xi) $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + C$

(xii) $\int \frac{-1}{\sqrt{a^2 - x^2}} dx = \cos^{-1}\left(\frac{x}{a}\right) + C$

(xiii) $\int \frac{1}{x\sqrt{x^2 - a^2}} dx = \frac{1}{a} \sec^{-1}\left(\frac{x}{a}\right) + C$

(xiv) $\int \frac{-1}{x\sqrt{x^2 - a^2}} dx = \frac{1}{a} \operatorname{cosec}^{-1}\left(\frac{x}{a}\right) + C$

(xv) $\int \sqrt{a^2 - x^2}\, dx = \frac{1}{2} x\sqrt{a^2 - x^2} + \frac{1}{2} a^2 \sin^{-1}\left(\frac{x}{a}\right) + C$

(xvi) $\int \sqrt{x^2 - a^2}\, dx = \frac{1}{2} x\sqrt{x^2 - a^2} - \frac{1}{2} a^2 \log|x + \sqrt{x^2 - a^2}| + C$

(xvii) $\int \sqrt{x^2 + a^2}\, dx = \frac{1}{2} x\sqrt{x^2 + a^2} + \frac{1}{2} a^2 \log|x + \sqrt{x^2 + a^2}| + C$

2. Trigonometric Formulae

(i) $\int \sin x\, dx = -\cos x + C$

(ii) $\int \cos x\, dx = \sin x + C$

(iii) $\int \tan x\, dx = -\log|\cos x| + C = \log|\sec x| + C$

(iv) $\int \cot x\, dx = \log|\sin x| + C = -\log|\operatorname{cosec} x| + C$

(v) $\int \sec x\, dx = \log|\sec x + \tan x| + C = \log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + C$

(vi) $\int \operatorname{cosec} x\, dx = \log|\operatorname{cosec} x - \cot x| + C = \log\left|\tan\frac{x}{2}\right| + C$

(vii) $\int \sec^2 x\, dx = \tan x + C$

(viii) $\int \operatorname{cosec}^2 x\, dx = -\cot x + C$

(ix) $\int \sec x \cdot \tan x\, dx = \sec x + C$

(x) $\int \operatorname{cosec} x \cdot \cot x\, dx = -\operatorname{cosec} x + C$

3. Exponential Formulae

(i) $\int e^x dx = e^x + C$

(ii) $\int e^{(ax+b)} dx = \frac{1}{a} \cdot e^{(ax+b)} + C$

(iii) $\int a^x\, dx = \frac{a^x}{\log_e a} + C,\ a > 0$ and $a \neq 1$

(iv) $\int a^{(bx+c)} dx = \frac{1}{b} \cdot \frac{a^{(bx+c)}}{\log_e a} + C,\ a > 0$ and $a \neq 1$

Methods of Integration

Following methods are used for integration

1. Integration by Substitutions

The method of reducing a given integral into one of the standard integrals, by a proper substitution, is called **method of substitution**.

To evaluate an integral of the form $\int f\{g(x)\} \cdot g'(x) dx$, we substitute $g(x) = t$, so that $g'(x) dx = dt$ and given integral reduces to $\int f(t)\, dt$.

NOTE
- $\int [f(x)]^n \cdot f'(x) = \frac{[f(x)]^{n+1}}{n+1} + C$
- If $\int f(x)\, dx = \phi(x)$, then $\int f(ax+b)\, dx = \frac{1}{a}\phi(ax+b) + C$

(i) To evaluate integrals of the form

$\int \frac{dx}{ax^2 + bx + c}$ or $\int \frac{dx}{\sqrt{ax^2 + bx + c}}$ or

$\int \sqrt{ax^2 + bx + c}\; dx$

We write, $ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$

$$= a\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4c}$$

This process reduces the integral to one of following forms

$$= \int \frac{dX}{X^2 - A^2}, \int \frac{dX}{X^2 + A^2} \text{ or } \int \frac{dX}{A^2 - X^2},$$

$$\int \frac{dX}{\sqrt{A^2 - X^2}}, \int \frac{dX}{\sqrt{X^2 - A^2}}, \int \frac{dX}{\sqrt{X^2 + A^2}}$$

or $\int \sqrt{A^2 - X^2} dX, \int \sqrt{X^2 - A^2}\, dX, \int \sqrt{A^2 + X^2}\, dX$

(ii) To evaluate integrals of the form

$\int \frac{(px+q)}{ax^2 + bx + c} dx$ or $\int \frac{(px+q)}{\sqrt{ax^2 + bx + c}} dx$

or $\int (px+q)\sqrt{ax^2 + bx + c}\; dx$

We put $px + q = A$ {differentiation of $(ax^2 + bx + c)$} + B, where A and B can be found by comparing the coefficients of like powers of x on the two sides.

2. Integration using Trigonometric Identities

In this method, we have to evaluate integrals of the form

- $\int \sin mx \cdot \cos nx\ dx$ or $\int \sin mx \cdot \sin nx\ dx$ or $\int \cos mx \cdot \cos nx\ dx$ or $\int \cos mx \cdot \sin nx\ dx$

In this method, we use the following trigonometrical identities

(i) $2\sin A \cdot \cos B = \sin(A+B) + \sin(A-B)$

(ii) $2\cos A \cdot \sin B = \sin(A+B) - \sin(A-B)$

(iii) $2\cos A \cdot \cos B = \cos(A+B) + \cos(A-B)$

(iv) $2\sin A \cdot \sin B = \cos(A-B) - \cos(A+B)$

(v) $2\sin A \cdot \cos A = \sin 2A$

(vi) $\cos^2 A = \left(\frac{1+\cos 2A}{2}\right)$

(vii) $\sin^2 A = \left(\frac{1-\cos 2A}{2}\right)$

(viii) $\cos^2 A - \sin^2 A = \cos 2A$

(ix) $\sin^2 A + \cos^2 A = 1$

3. Integration of Different Types of Functions

- To evaluate integrals of the form $\int \sin^p x \cos^q x\, dx$

Where $p, q \in Q$, we use the following rules :

(i) If p is odd, then put $\cos x = t$

(ii) If q is odd, then put $\sin x = t$

(iii) If both p, q are odd, then put either $\sin x = t$ or $\cos x = t$

(iv) If both p, q are even, then use trigonometric identities only.

(v) If p, q are rational numbers and $\left(\frac{p+q-2}{2}\right)$ is a negative integer, then put $\cot x = t$ or $\tan x = t$ as required.

- To evaluate integrals of the form $\int \frac{dx}{a+b\cos^2 x}$ or $\int \frac{dx}{a+b\sin^2 x}$ or $\int \frac{dx}{a\sin^2 x + b\cos^2 x}$, $\int \frac{dx}{(a\sin x + b\cos x)^2}$ or $\int \frac{dx}{a+b\sin^2 x + c\cos^2 x}$

(i) Divide both the numerator and denominator by $\cos^2 x$.

(ii) Replace $\sec^2 x$ by $1+\tan^2 x$ in the denominator, if any.

(iii) Put $\tan x = t$, so that $\sec^2 x\, dx = dt$

- To evaluate integrals of the form $\int \frac{1}{a\sin x + b\cos x} dx$ or $\int \frac{1}{a+b\sin x} dx$ or $\int \frac{1}{a+b\cos x} dx$ or $\int \frac{1}{a\sin x + b\cos x + c} dx$

(i) Put $\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ and $\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

(ii) Replace $1+\tan^2\frac{x}{2}$ by $\sec^2\frac{x}{2}$ and put $\tan\frac{x}{2} = t$.

- To evaluate integral of form $\int \frac{a\sin x + b\cos x}{c\sin x + d\cos x} dx$,

we write $a\sin x + b\cos x = A\frac{d}{dx}(c\sin x + d\cos x) + B(c\sin x + d\cos x)$

Where A and B can be found by equating the coefficient of $\sin x$ and $\cos x$ on both sides.

To evaluate integral of the form $\int \frac{a\sin x + b\cos x + c}{p\sin x + q\cos x + r} dx$.

We write $a\sin x + b\cos x + c = A\frac{d}{dx}(p\sin x + q\cos x + r) + B(p\sin x + q\cos x + r) + C$

Where A, B and C can be found by equating the coefficient of $\sin x$, $\cos x$ and the constant term.

- To evaluate integrals of the form $\int \frac{x^2+1}{x^4+kx^2+1} dx$ or $\int \frac{x^2-1}{x^4+kx^2+1} dx$

We divide the numerator and denominator by x^2 and make perfect square in denominator as $\left(x \pm \frac{1}{x}\right)^2$ and then put $x + \frac{1}{x} = t$ or $x - \frac{1}{x} = t$ as required.

- Substitution for Some Irrational Integrand

(i) $\sqrt{\frac{a-x}{a+x}}, \sqrt{\frac{a+x}{a-x}}$, $x = a\cos 2\theta$

(ii) $\sqrt{\frac{x}{a+x}}, \sqrt{\frac{a+x}{x}}, \sqrt{x(a+x)}, \frac{1}{\sqrt{x(a+x)}}$, $x = a\tan^2\theta$ or $x = a\cot^2\theta$

(iii) $\sqrt{\frac{x}{a-x}}, \sqrt{\frac{a-x}{x}}, \sqrt{x(a-x)}, \frac{1}{\sqrt{x(a-x)}}$, $x = a\sin^2\theta$ or $x = a\cos^2\theta$

(iv) $\sqrt{\frac{x}{x-a}}, \sqrt{\frac{x-a}{x}}, \sqrt{x(x-a)}, \frac{1}{\sqrt{x(x-a)}}$, $x = a\sec^2\theta$

(v) $\int \frac{dx}{(x-\alpha)(\beta-x)}, \int \sqrt{\left(\frac{x-\alpha}{\beta-x}\right)}dx$

$\int \sqrt{(x-\alpha)(\beta-x)}\,dx$, put $x = \alpha\cos^2\theta + \beta\sin^2\theta$

(vi) $\int \frac{dx}{(px+q)\sqrt{ax+b}}$, put $ax+b = t^2$

(vii) $\int \frac{dx}{(ax^2+bx+c)\sqrt{px+q}}$, put $px+q = t^2$

(viii) $\int \frac{dx}{(px+q)\sqrt{ax^2+bx+c}}$, put $px+q = \frac{1}{t}$

(ix) $\int \frac{dx}{(px^2+q)\sqrt{(ax^2+b)}}$ first put $x = \frac{1}{t}$

and then $a + bt^2 = z^2$

4. Integration by Parts

(i) If u and v are two functions of x, then

$$\int \underset{\text{I}}{u}\ \underset{\text{II}}{v}\,dx = u\int v\,dx - \int\left(\frac{du}{dx}\cdot\int v\,dx\right)dx$$

We use the following preference in order to select the first function

I → Inverse function
L → Logarithmic function
A → Algebraic function
T → Trigonometric function
E → Exponential function

(ii) If one of the function is not directly integrable, then we take it as the first function.

(iii) If both the functions are directly integrable, then the first function is chosen in such a way that its derivative vanishes easily or the function obtained in integral sign is easily integrable.

(iv) If only one which is not directly integrable, function is there e.g. $\int \log x\,dx$, then 1 (unity) is taken as second function.

Some more Special Integrals Based on Integration by Parts

(i) $\int e^x\{f(x) + f'(x)\}dx = f(x)e^x + C$

(ii) $\int e^{ax}\sin(bx+c)\,dx = \frac{e^{ax}}{a^2+b^2}\{a\sin(bx+c) - b\cos(bx+c)\} + k$

(iii) $\int e^{ax}\cos(bx+c)dx = \frac{e^{ax}}{a^2+b^2}\{a\cos(bx+c) + b\sin(bx+c)\} + k$

Here, c and k are integration constant.

5. Integration by Partial Fractions

To evaluate the integral of the form $\int \frac{P(x)}{Q(x)}dx$, where $P(x), Q(x)$ are polynomial in x with degree of $P(x) <$ degree of $Q(x)$ and $Q(x) \neq 0$, we use the method of partial fraction.

The partial fractions depend on the nature of the factors of $Q(x)$.

(i) According to nature of factors of $Q(x)$, corresponding form of partial fraction is given below:

If $Q(x) = (x-a_1)(x-a_2)(x-a_3)\ldots.(x-a_n)$, then we assume that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(x-a_1)} + \frac{A_2}{(x-a_2)} + \frac{A_3}{(x-a_3)} + \ldots + \frac{A_n}{(x-a_n)},$$

where the constants $A_1, A_2, \ldots, A_n$ can be determined by equating the coefficients of like power of x or by substituting $x = a_1, a_2, \ldots, a_n$.

(ii) If $Q(x) = (x-a)^k(x-a_1)(x-a_2)\ldots(x-a_r)$, then we assume that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(x-a)} + \frac{A_2}{(x-a)^2} + \ldots + \frac{A_k}{(x-a)^k} + \frac{B_1}{(x-a_1)} + \frac{B_2}{(x-a_2)} + \ldots + \frac{B_r}{(x-a_r)}$$

where the constants $A_1, A_2, \ldots, A_k, B_1, B_2 \ldots, B_r$ can be obtained by equating the coefficients of like power of x.

(iii) If some of the factors in $Q(x)$ are quadratic and non-repeating, corresponding to each quadratic factor ax^2+bx+c (non-factorisable), we assume the partial fraction of the type $\frac{Ax+B}{ax^2+bx+c}$, where A and B are constants to be determined by comparing coefficients of like powers of x.

(iv) If some of the factors in $Q(x)$ are quadratic and repeating, for every quadratic repeating factor of the type $(ax^2+bx+c)^k$ where ax^2+bx+c cannot be further factorise, we assume

$$\frac{A_1x + A_2}{ax^2+bx+c} + \frac{A_3x + A_4}{(ax^2+bx+c)^2} + \ldots + \frac{A_{2k-1}x + A_{2k}}{(ax^2+bx+c)^k}$$

If degree of $P(x) >$ degree of $Q(x)$, then we first divide $P(x)$ by $Q(x)$ so that $\frac{P(x)}{Q(x)}$ is expressed in the form of $T(x) + \frac{P_1(x)}{Q(x)}$, where $T(x)$ is a polynomial in x and $\frac{P_1(x)}{Q(x)}$ is a proper rational function

(i.e. degree of $P_1(x) <$ degree of $Q(x)$)

DAY PRACTICE SESSION 1

FOUNDATION QUESTIONS EXERCISE

1 If $\int \frac{dx}{x+x^7} = p(x)$, then $\int \frac{x^6}{x+x^7} dx$ is equal to → **JEE Mains 2013**

(a) $\log|x| - p(x) + C$ (b) $\log|x| + p(x) + C$
(c) $x - p(x) + C$ (d) $x + p(x) + C$

2 $\int \frac{x^3 - 1}{(x^4+1)(x+1)} dx$ is equal to

(a) $\frac{1}{4}\log(1+x^4) + \frac{1}{3}\log(1+x^3) + C$
(b) $\frac{1}{4}\log(1+x^4) - \frac{1}{3}\log(1+x^3) + C$
(c) $\frac{1}{4}\log(1+x^4) - \log(1+x) + C$
(d) $\frac{1}{4}\log(1+x^4) + \log(1+x) + C$

3 $\int (x+1)(x+2)^7(x+3)\, dx$ is equal to

(a) $\frac{(x+2)^{10}}{10} - \frac{(x+2)^8}{8} + C$
(b) $\frac{(x+1)^2}{2} - \frac{(x+2)^8}{8} - \frac{(x+3)^2}{2} + C$
(c) $\frac{(x+2)^{10}}{10} + C$
(d) $\frac{(x+1)^2}{2} + \frac{(x+2)^8}{8} + \frac{(x+3)^2}{2} + C$

4 The integral $\int \frac{dx}{x^2(x^4+1)^{\frac{3}{4}}}$ equals → **JEE Mains 2015**

(a) $\left(\frac{x^4+1}{x^4}\right)^{\frac{1}{4}} + C$ (b) $(x^4+1)^{\frac{1}{4}} + C$
(c) $-(x^4+1)^{\frac{1}{4}} + C$ (d) $-\left(\frac{x^4+1}{x^4}\right)^{\frac{1}{4}} + C$

5 If $\int \frac{\sin x}{\sin(x-\alpha)} dx = Ax + B\log\sin(x-\alpha) + C$, then the value of (A, B) is

(a) $(\sin\alpha, \cos\alpha)$ (b) $(\cos\alpha, \sin\alpha)$
(c) $(-\sin\alpha, \cos\alpha)$ (d) $(-\cos\alpha, \sin\alpha)$

6 If $\int \frac{f(x)}{\log\sin x} dx = \log\log\sin x + C$, then $f(x)$ is equal to

(a) $\sin x$ (b) $\cos x$
(c) $\log\sin x$ (d) $\cot x$

7 $\int \left\{\frac{(\log x - 1)}{1 + (\log x)^2}\right\}^2 dx$ is eqal to

(a) $\frac{x}{(\log x)^2 + 1} + C$ (b) $\frac{xe^x}{1+x^2} + C$
(c) $\frac{x}{x^2+1} + C$ (d) $\frac{\log x}{(\log x)^2 + 1} + C$

8 If $\int \sqrt{x + \sqrt{x^2+5}}\, dx = P\{x + \sqrt{x^2+5}\}^{3/2} + \frac{Q}{\sqrt{x\sqrt{x^2+5}}} + C$, then the value of $3PQ$ is

(a) -1 (b) -4 (c) -3 (d) -5

9 $\int \frac{dx}{\cos x - \sin x}$ is equal to

(a) $\frac{1}{\sqrt{2}}\log\left|\tan\left(\frac{x}{2} - \frac{\pi}{8}\right)\right| + C$ (b) $\frac{1}{\sqrt{2}}\log\left|\cot\left(\frac{x}{2}\right)\right| + C$
(c) $\frac{1}{\sqrt{2}}\log\left|\tan\left(\frac{x}{2} - \frac{3\pi}{8}\right)\right| + C$ (d) $\frac{1}{\sqrt{2}}\log\left|\tan\left(\frac{x}{2} + \frac{3\pi}{8}\right)\right| + C$

10 $\int \frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x \cos^2 x} dx$ is equal to

(a) $\sin 2x + C$ (b) $-\frac{1}{2}\sin 2x + C$
(c) $\frac{1}{2}\sin 2x + C$ (d) $-\sin 2x + C$

11 $\int \frac{(\sqrt[3]{x + \sqrt{2-x^2}})(\sqrt[6]{1 - x\sqrt{2-x^2}})}{\sqrt[3]{1-x^2}} dx; x \in (0,1)$ equals

(a) $2^{1/6}x + C$ (b) $2^{1/12}x + C$ (c) $2^{1/3}x + C$ (d) None of these

12 $\int \left(\frac{\cos 6x + 6\cos 4x + 15\cos 2x + 10}{10\cos^2 x + 5\cos x \cos 3x + \cos x \cos 5x}\right) dx = f(x) + C$, then $f(10)$ is equal to

(a) 20 (b) 10 (c) $2\sin 10$ (d) $2\cos 10$

13 The integral $\int \frac{2x^{12} + 5x^9}{(x^5 + x^3 + 1)^3} dx$ is equal to → **JEE Mains 2016**

(a) $\frac{-x^5}{(x^5+x^3+1)^2} + C$ (b) $\frac{x^{10}}{2(x^5+x^3+1)^2} + C$
(c) $\frac{x^5}{2(x^5+x^3+1)^2} + C$ (d) $\frac{-x^{10}}{2(x^5+x^3+1)^2} + C$

14 $\int \frac{x^2 - 1}{x^3\sqrt{2x^4 - 2x^2 + 1}} dx$ is equal to

(a) $\frac{\sqrt{2x^4 - 2x^2 + 1}}{x^2} + C$ (b) $\frac{\sqrt{2x^4 - 2x^2 + 1}}{x} + C$
(c) $\frac{\sqrt{2x^4 - 2x^2 + 1}}{2x} + C$ (d) $\frac{\sqrt{2x^4 - 2x^2 + 1}}{2x^2} + C$

15 $\int \frac{dx}{(1+x^2)\sqrt{p^2 + q^2(\tan^{-1}x)^2}}$ is equal to

(a) $\frac{1}{q}\log[q\tan^{-1}x + \sqrt{p^2 + q^2(\tan^{-1}x)^2}] + C$
(b) $\log[q\tan^{-1}x + \sqrt{p^2 + q^2(\tan^{-1}x)^2}] + C$
(c) $\frac{2}{3q}(p^2 + q^2\tan^{-1}x)^{3/2} + C$
(d) None of the above

16 In the integral $\int \frac{\cos 8x + 1}{\cot 2x - \tan 2x} dx = A \cos 8x + k$, where k is an arbitrary constant, then A is equal to

→ **JEE Mains 2013**

(a) $-\frac{1}{16}$ (b) $\frac{1}{16}$ (c) $\frac{1}{8}$ (d) $-\frac{1}{8}$

17 $\int \frac{(\sin\theta + \cos\theta)}{\sqrt{\sin 2\theta}} d\theta$ is equal to

→ **JEE Mains 2017**

(a) $\log|\cos\theta - \sin\theta + \sqrt{\sin 2\theta}|$
(b) $\log|\sin\theta - \cos\theta + \sqrt{\sin 2\theta}|$
(c) $\sin^{-1}(\sin\theta - \cos\theta) + C$
(d) $\sin^{-1}(\sin\theta + \cos\theta) + C$

18 $\int \left(\frac{f(x)\cdot g'(x) - f'(x)\cdot g(x)}{f(x)\cdot g(x)}\right)((\log g(x) - \log f(x))\,dx$ is equal to

(a) $\log\left(\frac{g(x)}{f(x)}\right) + C$ (b) $\frac{1}{2}\left(\frac{g(x)}{f(x)}\right)^2$

(c) $\frac{1}{2}\left(\log\left(\frac{g(x)}{f(x)}\right)\right)^2 + C$ (d) $\log\left(\frac{g(x)}{f(x)}\right)^2 + C$

19 Let $I_n = \int \tan^n x\, dx\ (n > 1)$. If $I_4 + I_6 = a\tan^5 x + bx^5 + C$, where C is a constant of integration, then the ordered pair (a, b) is equal to

(a) $\left(-\frac{1}{5}, 1\right)$ (b) $\left(\frac{1}{5}, 0\right)$ (c) $\left(\frac{1}{5}, -1\right)$ (d) $\left(-\frac{1}{5}, 0\right)$

20 If $\int \frac{\tan x}{1 + \tan x + \tan^2 x} dx = x - \frac{2}{\sqrt{A}} \tan^{-1}\left(\frac{2\tan x + 1}{\sqrt{A}}\right) + C$, then the value of A is

(a) 1 (b) 2
(c) 3 (d) None of these

21 $\int \cos^{-\frac{3}{7}} x \sin^{-\frac{11}{7}} x\, dx$ is equal to

(a) $\log|\sin^{\frac{4}{7}} x| + C$ (b) $\frac{4}{7}\tan^{\frac{4}{7}} x + C$

(c) $\frac{-7}{4}\tan^{-\frac{4}{7}} x + C$ (d) $\log|\cos^{\frac{3}{7}} x| + C$

22 $\int \frac{dx}{2 + \sin x + \cos x}$ is equal to

→ **NCERT Exemplar**

(a) $\sqrt{2}\tan^{-1}\left(\frac{\tan(x/2) + 1}{\sqrt{2}}\right) + C$ (b) $\tan^{-1}\left(\frac{\tan(x/2) + 1}{\sqrt{2}}\right) + C$

(c) $\sqrt{2}\tan^{-1}\left(\frac{\tan(x/2)}{\sqrt{2}}\right) + C$ (d) None of these

23 If the integral $\int \frac{5\tan x}{\tan x - 2} dx = x + a\ln|\sin x - 2\cos x| + k$, then a is equal to

(a) -1 (b) -2
(c) 1 (d) 2

24 $\int \frac{x^2 - 1}{x^4 + x^2 + 1} dx$ is equal to

(a) $\frac{1}{2}\log\left(\frac{x^2 + x + 1}{x^2 - x + 1}\right) + C$ (b) $\frac{1}{2}\log\left(\frac{x^2 - x - 1}{x^2 + x + 1}\right) + C$

(c) $\log\left(\frac{x^2 - x + 1}{x^2 + x + 1}\right) + C$ (d) $\frac{1}{2}\log\left(\frac{x^2 - x + 1}{x^2 + x + 1}\right) + C$

25 $\int \sqrt{\frac{x}{a^3 - x^3}} dx$ is equal to

(a) $\sin^{-1}\left(\frac{x}{a}\right)^{3/2} + C$ (b) $\frac{2}{3}\sin^{-1}\left(\frac{x}{a}\right)^{3/2} + C$

(c) $\frac{3}{2}\sin^{-1}\left(\frac{x}{a}\right)^{3/2} + C$ (d) $\frac{3}{2}\sin^{-1}\left(\frac{x}{a}\right)^{2/3} + C$

26 If an anti-derivative of $f(x)$ is e^x and that of $g(x)$ is $\cos x$, then $\int f(x)\cos x\, dx + \int g(x) e^x dx$ is equal to

(a) $f(x)\cdot g(x) + C$ (b) $f(x) + g(x) + C$
(c) $e^x \cos x + C$ (d) $f(x) - g(x) + C$

27. If $\int f(x)\,dx = \Psi(x)$, then $\int x^5 f(x^3)\,dx$ is equal to

→ **JEE Mains 2013**

(a) $\frac{1}{3}[x^3\Psi(x^3)] - \int x^2\Psi(x^3)dx + C$
(b) $\frac{1}{3}[x^3\Psi(x^3)] - 3\int x^3\Psi(x^3)dx + C$
(c) $\frac{1}{3}[x^3\Psi(x^3) - \int x^2\Psi(x^3)dx] + C$
(d) $\frac{1}{3}[x^3\Psi(x^3)] - \int x^3\Psi(x^3)dx + C$

28 If $\int \frac{1 - 6\cos^2 x}{\sin^6 x \cos^2 x} dx = \frac{f(x)}{(\sin x)^6} + C$, then $f(x)$ is equal to

(a) $\sin x$ (b) $\cos x$ (c) $\tan x$ (d) $\cot x$

29 $\int \tan^{-1}\sqrt{x}\,dx$ is equal to

→ **NCERT Exemplar**

(a) $(x + 1)\tan^{-1}\sqrt{x} - \sqrt{x} + C$ (b) $x\tan^{-1}\sqrt{x} - \sqrt{x} + C$
(c) $\sqrt{x} - x\tan^{-1}\sqrt{x} + C$ (d) $\sqrt{x} - (x + 1)\tan^{-1}\sqrt{x} + C$

30 If $I_n = \int (\log x)^n dx$, then $I_n + n I_{n-1}$ is equal to

(a) $x(\log x)^n$ (b) $(x\log x)^n$ (c) $(\log x)^{n-1}$ (d) $n(\log x)^n$

31 If $\int f(x)\,dx = g(x)$, then $\int f^{-1}(x)\,dx$ is equal to

(a) $g^{-1}(x)$ (b) $xf^{-1}(x) - g(f^{-1}(x))$
(c) $xf^{-1}(x) - g^{-1}(x)$ (d) $f^{-1}(x)$

32 $\int \frac{(x + 3)e^x}{(x + 4)^2} dx$ is equal to

(a) $\frac{1}{(x + 4)^2} + C$ (b) $\frac{e^x}{(x + 4)^2} + C$

(c) $\frac{e^x}{x + 4} + C$ (d) $\frac{e^x}{x + 3} + C$

33 If $\int \frac{x^2 - x + 1}{x^2 + 1} e^{\cot^{-1} x} dx = A(x)\, e^{\cot^{-1} x} + C$, then $A(x)$ is equal to

→ **JEE Mains 2013**

(a) $-x$ (b) x (c) $\sqrt{1 - x}$ (d) $\sqrt{1 + x}$

34 If $g(x)$ is a differentiable function satisfying $\frac{d}{dx}\{g(x)\} = g(x)$ and $g(0) = 1$, then $\int g(x)\left(\frac{2-\sin 2x}{1-\cos 2x}\right) dx$ is equal to

(a) $g(x) \cot x + C$ (b) $-g(x) \cot x + C$

(c) $\frac{g(x)}{1-\cos 2x} + C$ (d) None of these

35 $\int \frac{dx^3}{x^3(x^n+1)}$ is equal to

(a) $\frac{3}{n}\ln\left(\frac{x^n}{x^n+1}\right) + C$ (b) $\frac{1}{n}\ln\left(\frac{x^n}{x^n+1}\right) + C$

(c) $\frac{3}{n}\ln\left(\frac{x^n+1}{x^n}\right) + C$ (d) $3n\ln\left(\frac{x^{n+1}}{x^n}\right) + C$

DAY PRACTICE SESSION 2

PROGRESSIVE QUESTIONS EXERCISE

1 If $\int f(x)\,dx = f(x)$, then $\int \{f(x)\}^2\,dx$ is equal to

(a) $\frac{1}{2}\{f(x)\}^2$ (b) $\{f(x)\}^3$ (c) $\frac{\{f(x)\}^3}{3}$ (d) $\{f(x)\}^2$

2 If $f(x) = \begin{vmatrix} 0 & x^2-\sin x & \cos x - 2 \\ \sin x - x^2 & 0 & 1-2x \\ 2-\cos x & 2x-1 & 0 \end{vmatrix}$,

then $\int f(x)\,dx$ is equal to

(a) $\frac{x^3}{3} - x^2 \sin x + \sin 2x + C$

(b) $\frac{x^3}{3} - x^2 \sin x - \cos 2x + C$

(c) $\frac{x^3}{3} - x^2 \cos x - \cos 2x + C$

(d) None of the above

3 $\int e^{2ax}\frac{1-\cos 2a\,x}{1+\sin 2a\,x}\,dx$ is equal to

(a) $-\frac{1}{a}e^{2ax}\cos\left(\frac{\pi}{4}+ax\right)+C$

(b) $-\frac{1}{2a}e^{2ax}\cot\left(\frac{\pi}{4}+ax\right)+C$

(c) $-\frac{1}{2a}e^{2ax}\cos\left(\frac{\pi}{4}+ax\right)+C$

(d) $-\frac{1}{a}e^{2ax}\operatorname{cosec}\left(\frac{\pi}{4}+ax\right)+C$

4 If $x^2 \neq n\pi - 1$, $n \in N$. Then, the value of

$$\int x\sqrt{\frac{2\sin(x^2+1)-\sin 2(x^2+1)}{2\sin(x^2+1)+\sin 2(x^2+1)}}\,dx \text{ is equal to}$$

(a) $\log\left|\frac{1}{2}\sec(x^2+1)\right| + C$ (b) $\log\left|\sec\left(\frac{x^2+1}{2}\right)\right| + C$

(c) $\frac{1}{2}\log|\sec(x^2+1)| + C$ (d) None of these

5 The integral

$$\int \frac{\sin^2 x \cos^2 x}{(\sin^5 x + \cos^3 x \sin^2 x + \sin^3 x \cos^2 x + \cos^5 x)^2}\,dx$$

is equal to → **JEE Mains 2018**

(a) $\frac{1}{3(1+\tan^3 x)} + C$ (b) $\frac{-1}{3(1+\tan^3 x)} + C$

(c) $\frac{1}{1+\cot^3 x} + C$ (d) $\frac{-1}{1+\cot^3 x} + C$

(where C is a constant of integration)

6 $\int \cos 2\theta \log\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right) d\theta$ is equal to

(a) $(\cos\theta-\sin\theta)^2\log\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)+C$

(b) $(\cos\theta+\sin\theta)^2\log\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)+C$

(c) $\frac{(\cos\theta-\sin\theta)^2}{2}\log\left(\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}\right)+C$

(d) $\frac{1}{2}\sin 2\theta \log\tan\left(\frac{\pi}{4}+\theta\right)-\frac{1}{2}\log\sec 2\theta + C$

7 $\int \frac{x^2}{(x\sin x + \cos x)^2}\,dx$ is equal to

(a) $\frac{\sin x + \cos x}{x\sin x + \cos x} + C$

(b) $\frac{x\sin x - \cos x}{x\sin x + \cos x} + C$

(c) $\frac{\sin x - x\cos x}{x\sin x + \cos x} + C$

(d) None of the above

8 If $f(x) = \int \frac{x^2\,dx}{(1+x^2)(1+\sqrt{1+x^2})}$ and $f(0) = 0$, then the value of $f(1)$ is

(a) $\log(1+\sqrt{2})$ (b) $\log(1+\sqrt{2})-\frac{\pi}{4}$

(c) $\log(1+\sqrt{2})+\frac{\pi}{2}$ (d) None of these

9 If $I = \int \frac{dx}{\sqrt[3]{(x+1)^2(x-1)^4}} = k\sqrt[3]{\frac{1+x}{1-x}} + C$, then k is equal to

(a) 2/3 (b) 3/2

(c) 1/3 (d) 1/2

10 $\int \frac{dx}{(\sin x+2)(\sin x-1)}$ is equal to

(a) $\frac{2}{3\left(\tan\frac{x}{2}-1\right)}-\frac{2}{3\sqrt{3}}\tan^{-1}\left[\frac{2\left(\tan\frac{x}{2}+\frac{1}{2}\right)}{\sqrt{3}}\right]+C$

(b) $\frac{2}{\left(\tan\frac{x}{2}+1\right)}+C$

(c) $-\frac{2}{3}\frac{1}{\left(\tan\frac{x}{2}-1\right)}+\frac{2}{3\sqrt{3}}\tan^{-1}\left[\frac{2\tan\frac{x}{2}-1}{\sqrt{3}}\right]+C$

(d) $-\frac{2}{3}\frac{2}{\left(\tan\frac{x}{2}-1\right)}+\frac{2}{3\sqrt{3}}\tan^{-1}\left[\frac{2\tan\frac{x}{2}-1}{\sqrt{3}}\right]+C$

11 $\int \frac{x^2}{(2+3x^2)^{5/2}}dx$ is equal to

(a) $\frac{1}{5}\left[\frac{x^2}{2+3x^2}\right]^{3/2}+C$

(b) $\frac{1}{6}\left[\frac{x^2}{2+3x^2}\right]^{3/2}+C$

(c) $\frac{1}{6}\left[\frac{x^2}{2+3x^2}\right]^{7/2}+C$

(d) None of the above

12 The integral $\int\left(1+x-\frac{1}{x}\right)e^{x+\frac{1}{x}}dx$ is equal to

→ **JEE Mains 2014**

(a) $(x-1)e^{x+\frac{1}{x}}+C$ (b) $xe^{x+\frac{1}{x}}+C$

(c) $(x+1)e^{x+\frac{1}{x}}+C$ (d) $-xe^{x+\frac{1}{x}}+C$

13 $\int(\sin(101x)\cdot\sin^{99}x)\,dx$ is equal to

(a) $\frac{\sin(100x)(\sin x)^{100}}{100}+C$ (b) $\frac{\cos(100x)(\sin x)^{100}}{100}+C$

(c) $\frac{\cos(100x)(\cos x)^{100}}{100}+C$ (d) $\frac{\cos(100x)(\cos x)^{100}}{100}+C$

14 $\int\sqrt{\frac{(2018)^{2x}}{1-(2018)^{2x}}}\,(2018)^{\sin^{-1}(2018)^x}dx$ is equal to

(a) $(\log_{2018}e)^2(2018)^{\sin^{-1}(2018)^x}+C$

(b) $(\log_{2018}e)^2(2018)^{x+\sin^{-1}(2018)^x}+C$

(c) $(\log_{2018}e)^2(2018)^{x-\sin^{-1}(2018)^x}+C$

(d) $\frac{(2018)^{\sin^{-1}(2018)^x}}{(\log_{2018}e)^2}+C$

15 $\int\left(\int e^x\left(\log x+\frac{2}{x}-\frac{1}{x^2}\right)\right)dx$ is equal to

(a) $e^x\log x+C_1x+C_2$ (b) $\log x+\frac{1}{x}+C_1x+C_2$

(c) $\frac{\log x}{x}+C_1x+C_2$ (d) None of these

ANSWERS

SESSION 1

1 (a)	**2** (c)	**3** (a)	**4** (d)	**5** (b)	**6** (d)	**7** (a)	**8** (d)	**9** (d)	**10** (b)
11 (a)	**12** (a)	**13** (b)	**14** (d)	**15** (a)	**16** (a)	**17** (c)	**18** (c)	**19** (b)	**20** (c)
21 (c)	**22** (a)	**23** (d)	**24** (d)	**25** (b)	**26** (c)	**27** (a)	**28** (c)	**29** (a)	**30** (a)
31 (b)	**32** (c)	**33** (b)	**34** (b)	**35** (a)					

SESSION 2

1 (a)	**2** (d)	**3** (b)	**4** (b)	**5** (b)	**6** (d)	**7** (c)	**8** (b)	**9** (b)	**10** (a)
11 (b)	**12** (b)	**13** (a)	**14** (a)	**15** (a)					

Hints and Explanations

1 Let $I = \int \frac{x^6}{x + x^7}dx = \int \frac{x^6}{x(1 + x^6)}dx$

$$= \int \frac{(1 + x^6) - 1}{x(1 + x^6)}dx$$

$$\Rightarrow I = \int \frac{dx}{x} - \int \frac{dx}{x + x^7}$$

$$= \log|x| - p(x) + C$$

2 Let $I = \int \frac{x^3 - 1}{(x^4 + 1)(x + 1)}dx$

$$= \int \frac{x^3 + x^4 - x^4 - 1}{(x^4 + 1)(x + 1)}dx$$

$$= \int \frac{x^3(x + 1) - (x^4 + 1)}{(x^4 + 1)(x + 1)}dx$$

$$= \int \left[\frac{x^3}{x^4 + 1} - \frac{1}{x + 1}\right]dx$$

$$= \frac{1}{4}\log(x^4 + 1) - \log(x + 1) + C$$

3 Let $I = \int (x + 1)(x + 2)^7(x + 3)dx$

Put $x + 2 = t$

$\Rightarrow \quad x = t - 2$ and $dx = dt$

$\therefore \quad I = \int (t - 1)t^7(t + 1)dx$

$$= \int (t^2 - 1)\cdot t^7\, dx = \int (t^9 - t^7)dx$$

$$= \frac{t^{10}}{10} - \frac{t^8}{8} + C = \frac{(x + 2)^{10}}{10} - \frac{(x + 2)^8}{8} + C$$

4 $\int \frac{dx}{x^2(x^4 + 1)^{\frac{3}{4}}} = \int \frac{dx}{x^5\left(1 + \frac{1}{x^4}\right)^{\frac{3}{4}}}$

Put $1 + \frac{1}{x^4} = t^4$

$$\Rightarrow \quad -\frac{4}{x^5}dx = 4t^3dt$$

$$\Rightarrow \quad \frac{dx}{x^5} = -t^3dt$$

Hence, the integral becomes

$$\int \frac{-t^3dt}{t^3} = -\int dt = -t + C$$

$$= -\left(1 + \frac{1}{x^4}\right)^{1/4} + C = -\left(\frac{x^4 + 1}{x^4}\right)^{1/4} + C$$

5 Let $I = \int \frac{\sin x}{\sin(x - \alpha)}dx$

Put $x - \alpha = t \Rightarrow dx = dt$

$$\therefore I = \int \frac{\sin(t + \alpha)}{\sin t}dt$$

$$= \int \cos\alpha\, dt + \int \sin\alpha \cdot \frac{\cos t}{\sin t}dt$$

$$= \cos\alpha \cdot t + \sin\alpha \log \sin t + C$$

$$= x\cos\alpha + \sin\alpha \log\{\sin(x - \alpha)\} + C$$

$\therefore \quad A = \cos\alpha,\ B = \sin\alpha$

6 We have, $\int \frac{f(x)}{\log \sin x} = \log(\log \sin x) + C$

$$\therefore f(x) = \frac{d}{dx}(\log \sin x) \quad \left[\because \int \frac{f'(x)}{f(x)}dx = \log(f(x)) + C\right]$$

$$= \cot x$$

7 $\int \left\{\frac{(\log x - 1)}{1 + (\log x)^2}\right\}^2 dx$

$$= \int \frac{(\log x)^2 + 1 - 2\log x}{[(\log x)^2 + 1]^2}dx$$

$$= \int \frac{(\log x)^2 + 1 - 2x\left(\log x \cdot \frac{1}{x}\right)}{[(\log x)^2 + 1]^2}dx$$

$$= \int \frac{d}{dx}\left[\frac{x}{(\log x)^2 + 1}\right]dx$$

$$= \frac{x}{(\log x)^2 + 1} + C$$

8 Let $I = \int \sqrt{x + \sqrt{x^2 + 5}}\,dx$

Put $x + \sqrt{x^2 + 5} = t$

$\Rightarrow \quad \sqrt{x^2 + 5} = t - x$

$\Rightarrow \quad x^2 + 5 = t^2 + x^2 - 2xt$

$\Rightarrow \quad 5 = t^2 - 2xt$

$\Rightarrow \quad 2xt = t^2 - 5$

$\Rightarrow \quad x = \frac{1}{2}\left(t - \frac{5}{t}\right)$

and $\quad dx = \frac{1}{2}\left(1 + \frac{5}{t^2}\right)$

Now, $\quad I = \int t^{1/2}\cdot\frac{1}{2}\left(1 + \frac{5}{t^2}\right)dt$

$$= \frac{1}{2}\int (t^{1/2} + 5t^{-3/2})dt$$

$$= \frac{1}{2}\left(\frac{2}{3}t^{3/2} - \frac{10}{\sqrt{t}}\right) + C = \frac{1}{3}t^{3/2} - \frac{5}{\sqrt{t}} + C$$

Clearly, $3PQ = -5$

9 Let $I = \frac{1}{\sqrt{2}}\int \frac{dx}{\left(\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x\right)}$

$$= \frac{1}{\sqrt{2}}\int \sec\left(x + \frac{\pi}{4}\right)dx$$

$$= \frac{1}{\sqrt{2}}\log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2} + \frac{\pi}{8}\right)\right| + C$$

$$= \frac{1}{\sqrt{2}}\log\left|\tan\left(\frac{x}{2} + \frac{3\pi}{8}\right)\right| + C$$

10 Let $\quad I = \int \frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x\cos^2 x}dx$

$$I = \int \frac{(\sin^4 x + \cos^4 x)(\sin^4 x - \cos^4 x)}{(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x}dx$$

$$= \int (\sin^4 x - \cos^4 x)dx$$

$$= \int (\sin^2 x - \cos^2 x)dx$$

$$= \int -\cos 2x\, dx$$

$$= -\frac{\sin 2x}{2} + C$$

11 Let $I = \int \frac{(\sqrt[3]{x + \sqrt{2 - x^2}})(\sqrt[6]{1 - x\sqrt{2 - x^2}})}{\sqrt[3]{1 - x^2}}dx$

$$= \int \frac{\sqrt[3]{x + \sqrt{2 - x^2}}\left(\sqrt[6]{\frac{1}{2}(2 - 2x\sqrt{2 - x^2})}\right)}{\sqrt[3]{1 - x^2}}dx$$

$$= \int \frac{\sqrt[3]{x + \sqrt{2 - x^2}}\left(\sqrt[6]{\frac{[x^2 + (\sqrt{2 - x^2})^2 - 2x\sqrt{2 - x^2}]}{2}}\right)}{\sqrt[3]{1 - x^2}}dx$$

$$= \int \frac{\sqrt[3]{x + \sqrt{2 - x^2}}\sqrt[3]{\sqrt{2 - x^2} - x}}{2^{1/6}\sqrt[3]{1 - x^2}}dx$$

$$= \int \frac{\sqrt[3]{(2 - x^2) - x^2}}{2^{1/6}\sqrt[3]{1 - x^2}}dx$$

$$= 2^{1/6}\int dx = 2^{1/6}x + C$$

12 Let $\quad I = \int \left(\frac{\cos 6x + 6\cos 4x + 15\cos 2x + 10}{10\cos^2 x + 5\cos x\cos 3x + \cos x\cos 5x}\right)dx$

$$= \int \frac{(\cos 6x + \cos 4x) + 5(\cos 4x + \cos 2x) + 10(\cos 2x + 1)}{10\cos^2 x + 5\cos x\cos 3x + \cos x\cos 5x}dx$$

$$= \int \frac{2\cos 5x\cdot\cos x + 10\cdot\cos 3x\cdot\cos x + 10\cdot(2\cos^2 x)}{10\cos^2 x + 5\cos x\cdot\cos 3x + \cos x\cos 5x}dx = 2\int dx$$

$$= 2x + C$$

Clearly, $f(10) = 20$

13 Let $I = \int \frac{2x^{12} + 5x^9}{(x^5 + x^3 + 1)^3}dx$

$$= \int \frac{2x^{12} + 5x^9}{x^{15}(1 + x^{-2} + x^{-5})^3}dx$$

$$= \int \frac{2x^{-3} + 5x^{-6}}{(1 + x^{-2} + x^{-5})^3}dx$$

Now, put $1 + x^{-2} + x^{-5} = t$

$\Rightarrow \quad (-2x^{-3} - 5x^{-6})dx = dt$

$\Rightarrow \quad (2x^{-3} + 5x^{-6})dx = -dt$

$$\therefore \quad I = -\int \frac{dt}{t^3} = -\int t^{-3}dt$$

$$= -\frac{t^{-3+1}}{-3+1} + C = \frac{1}{2t^2} + C$$

$$= \frac{x^{10}}{2(x^5 + x^3 + 1)^2} + C$$

14 Let $I = \int \frac{x^2 - 1}{x^3\sqrt{2x^4 - 2x^2 + 1}}dx$

$$= \int \frac{x^2 - 1}{x^5\sqrt{2 - \frac{2}{x^2} + \frac{1}{x^4}}}dx$$

$$= \int \frac{\frac{1}{x^3} - \frac{1}{x^5}}{\sqrt{2 - \frac{2}{x^2} + \frac{1}{x^4}}}dx$$

Now, putting $2 - \frac{2}{x^2} + \frac{1}{x^4} = t$, we get

$$4\left(\frac{1}{x^3} - \frac{1}{x^5}\right)dx = dt$$

$$\therefore\ I = \frac{1}{4}\int \frac{dt}{\sqrt{t}}$$

$$= \frac{1}{4}\cdot 2\sqrt{t} + C = \frac{\sqrt{2x^4 - 2x^2 + 1}}{2x^2} + C$$

15 Put $q\tan^{-1} x = t$

$$\Rightarrow \frac{q}{1 + x^2}dx = dt \Rightarrow \frac{1}{1 + x^2}dx = \frac{dt}{q}$$

$$\therefore \int \frac{dt}{q\sqrt{p^2 + t^2}} = \frac{1}{q}\log\,[t + \sqrt{p^2 + t^2}]$$

$$= \frac{1}{q}\log\,[q\tan^{-1} x + \sqrt{p^2 + q^2(\tan^{-1} x)^2}] + C$$

16 LHS $= \int \frac{2\cos^2 4x}{\frac{\cos^2 2x - \sin^2 2x}{\cos 2x \sin 2x}}dx$

$$= \int \frac{2\cos^2 4x \times \cos 2x \sin 2x}{\cos 4x}dx$$

$$= \int \cos 4x \times \sin 4x dx = \frac{1}{2}\int \sin 8x dx$$

$$= \frac{-1}{2}\frac{\cos 8x}{8} + k$$

Hence, we get $A = \frac{-1}{16}$

17 Let $I = \int \frac{\sin\theta + \cos\theta}{\sqrt{1 - (1 - 2\sin\theta\cos\theta)}}d\theta$

$$= \int \frac{\sin\theta + \cos\theta}{\sqrt{1 - (\sin\theta - \cos\theta)^2}}d\theta$$

Put $\sin\theta - \cos\theta = t$

$\Rightarrow (\cos\theta + \sin\theta)\,d\theta = dt$

$$\therefore\ I = \int \frac{dt}{\sqrt{1 - t^2}} = \sin^{-1}(t) + C$$

$$= \sin^{-1}(\sin\theta - \cos\theta) + C$$

18 Let $I = \int \left(\frac{f(x)\cdot g'(x) - f'(x)\cdot g(x)}{f(x)\cdot g(x)}\right)\log\left(\frac{g(x)}{f(x)}\right)dx$

Put $\frac{g(x)}{f(x)} = t$

$$\Rightarrow \frac{f(x)\cdot g'(x) - g(x)\cdot f'(x)}{(f(x))^2}dx = dt$$

$$\Rightarrow \frac{f(x)\cdot g'(x) - g(x)\cdot f'(x)}{f(x)\cdot g(x)}\cdot\frac{g(x)}{f(x)}\cdot dx = dt$$

$$\Rightarrow \left(\frac{f(x)\cdot g'(x) - g(x)\cdot f'(x)}{f(x)\cdot g(x)}\right)dx = \frac{dt}{t}$$

Now, $I = \int \frac{1}{t}\cdot\log t\,dt = \frac{(\log t)^2}{2} + C$

$$= \frac{1}{2}\left(\log\left(\frac{g(x)}{f(x)}\right)\right)^2 + C$$

19 We have, $I_n = \int \tan^n x\,dx$

$$\therefore\ I_n + I_{n+2} = \int \tan^n x\,dx + \int \tan^{n+2} x\,dx$$

$$= \int \tan^n x(1 + \tan^2 x)\,dx$$

$$= \int \tan^n x \sec^2 x\,dx$$

$$= \frac{\tan^{n+1} x}{n + 1} + C$$

Put $n = 4$, we get $I_4 + I_6 = \frac{\tan^5 x}{5} + C$

$\therefore\ a = \frac{1}{5}$ and $b = 0$

20 Let $I = \int \frac{\tan x}{1 + \tan x + \tan^2 x}$

$$dx = \int \frac{\frac{\sin x}{\cos x}}{1 + \frac{\sin^2 x}{\cos^2 x} + \frac{\sin x}{\cos x}}dx$$

$$= \int \frac{\sin 2x}{2 + \sin 2x}dx$$

$$= \int dx - 2\int \frac{dx}{2 + \sin 2x}$$

$$= x - 2\int \frac{\sec^2 x}{2\sec^2 x + 2\tan x}dx$$

Let $\tan x = t$

$\Rightarrow \sec^2 x\,dx = dt$

$$= x - \frac{2}{2}\int \frac{dt}{t^2 + t + 1}$$

$$= x - \int \frac{dt}{\left(t + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$\Rightarrow I = x - \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan x + 1}{\sqrt{3}}\right) + C$$

Hence, we get $A = 3$.

21 Here, $m + n = \frac{-3}{7} + \left(\frac{-11}{7}\right) = -2$

$$I = \int \cos^{-3/7} x(\sin^{(-2+3/7)} x)dx$$

$$= \int \cos^{-3/7} x \sin^{-2} x \sin^{3/7} x dx$$

$$= \int \frac{\operatorname{cosec}^2 x}{\left(\frac{\cos^{3/7} x}{\sin^{3/7} x}\right)}dx = \int \frac{\operatorname{cosec}^2 x}{\cot^{3/7} x}dx$$

Put $\cot x = t \Rightarrow -\operatorname{cosec}^2 x\,dx = dt$

$$\therefore\ I = -\int \frac{dt}{t^{3/7}} = \frac{-7}{4}t^{4/7} + C$$

$$= -\frac{7}{4}\tan^{-4/7} x + C$$

22 Let $I = \int \frac{dx}{2 + \sin x + \cos x}$

$$\Rightarrow I = \int \frac{dx}{2 + \frac{2\tan\frac{x}{2}}{1 + \tan^2\frac{x}{2}} + \frac{1 - \tan^2\frac{x}{2}}{1 + \tan^2\frac{x}{2}}}$$

$$= \int \frac{\sec^2\frac{x}{2}dx}{2 + 2\tan^2\frac{x}{2} + 2\tan\frac{x}{2} + 1 - \tan^2\frac{x}{2}}$$

$$I = \int \frac{\sec^2\frac{x}{2}dx}{\tan^2\frac{x}{2} + 2\tan\frac{x}{2} + 3}$$

Put $\tan\frac{x}{2} = t \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}dx = dt$

$$\therefore I = \int \frac{2dt}{t^2 + 2t + 3}$$

$$= 2\int \frac{dt}{t^2 + 2t + 1 + 2} = 2\int \frac{2dt}{(t + 1)^2 + (\sqrt{2})^2}$$

$$= 2\cdot\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t + 1}{\sqrt{2}}\right) + C$$

$$\Rightarrow I = \sqrt{2}\tan^{-1}\left(\frac{\tan\frac{x}{2} + 1}{\sqrt{2}}\right) + C$$

23 Given, $\int \frac{5\tan x}{\tan x - 2}dx$

$= x + a\ln|\sin x - 2\cos x| + k$ ...(i)

Now, let us assume that

$$I = \int \frac{5\tan x}{\tan x - 2}dx$$

On multiplying by $\cos x$ in numerator and denominator, we get

$$I = \int \frac{5\sin x}{\sin x - 2\cos x}dx$$

Let $5\sin x = A(\sin x - 2\cos x) + B(\cos x + 2\sin x)$

$\Rightarrow 0\cdot\cos x + 5\sin x = (A + 2B)\sin x + (B - 2A)\cos x$

On comparing the coefficients of $\sin x$ and $\cos x$, we get

$A + 2B = 5$ and $B - 2A = 0$

$\Rightarrow\ A = 1$ and $B = 2$

$\Rightarrow\ 5\sin x = (\sin x - 2\cos x) + 2(\cos x + 2\sin x)$

$$\Rightarrow I = \int \frac{5\sin x}{\sin x - 2\cos x}dx$$

$$= \int \frac{(\sin x - 2\cos x) + 2(\cos x + 2\sin x)}{(\sin x - 2\cos x)}dx$$

$$\Rightarrow \quad I = \int 1\,dx + 2\int \frac{d(\sin x - 2\cos x)}{(\sin x - 2\cos x)}$$

$$I = x + 2\log|(\sin x - 2\cos x)| + k \qquad \ldots\text{(ii)}$$

where, k is the constant of integration.
On comparing the value of I in Eqs. (i) and (ii), we get $a = 2$

24 $\displaystyle\int \frac{\left(1 - \frac{1}{x^2}\right)}{\left(x + \frac{1}{x}\right)^2 - 1}dx$

Put $x + \frac{1}{x} = t \Rightarrow \left(1 - \frac{1}{x^2}\right)dx = dt$

$$\therefore \int \frac{dt}{t^2 - 1} = \frac{1}{2}\log\left(\frac{t-1}{t+1}\right) + C$$

$$= \frac{1}{2}\log\left(\frac{x^2 - x + 1}{x^2 + x + 1}\right) + C$$

25 Let $I = \displaystyle\int \sqrt{\frac{x}{a^3 - x^3}}\,dx$

Put $x = a(\sin\theta)^{2/3}$

$$\Rightarrow \quad dx = \frac{2}{3}a(\sin\theta)^{-1/3}\cos\theta\, d\theta$$

$$a^{1/2}(\sin\theta)^{1/3}\cdot\frac{2}{3}$$

$$\therefore \quad I = \int \frac{a(\sin\theta)^{-1/3}\cdot\cos\theta}{\sqrt{a^3 - a^3\sin^2\theta}}d\theta$$

$$= \frac{2}{3}\int \frac{a^{3/2}\cdot\cos\theta}{a^{3/2}\cos\theta}d\theta = \frac{2}{3}\int d\theta$$

$$= \frac{2}{3}\theta + C = \frac{2}{3}\sin^{-1}\left(\frac{x}{a}\right)^{3/2} + C$$

26 $\displaystyle\int f(x)\cos x\,dx + \int g(x)e^x dx$

$$= \frac{e^x}{2}(\cos x + \sin x) - \frac{e^x}{2}(\sin x - \cos x) + C$$

$$= \frac{e^x}{2}(2\cos x) + C$$

$$= e^x\cos x + C$$

27 Given, $\int f(x)\,dx = \Psi(x)$

Let $\quad I = \int x^5 f(x^3)\,dx$

Put $\quad x^3 = t$

$\Rightarrow \quad x^2dx = \dfrac{dt}{3} \quad \ldots$(i)

$$\therefore I = \frac{1}{3}\int t\,f(t)\,dt = \frac{1}{3}[t\,\Psi(t) - \int\Psi(t)\,dt]$$

$$= \frac{1}{3}[x^3\Psi(x^3) - 3\int x^2\Psi(x^3)\,dx] + C$$

[from Eq. (i)]

$$= \frac{1}{3}x^3\Psi(x^3) - \int x^2\Psi(x^3)\,dx + C$$

28 Let $I = \displaystyle\int \frac{1 - 6\cos^2 x}{\sin^6 x\cos^2 x}dx$

$$= \int \frac{1}{\sin^6 x\cos^2 x}dx - 6\int\frac{dx}{\sin^6 x}$$

$$= I_1 - I_2 \qquad \text{(say)}$$

Here, $I_1 = \displaystyle\int \frac{\sec^2 x}{\sin^6 x}dx$

$$= \int \frac{1}{\sin^6 x}\cdot\sec^2 x\,dx = \frac{1}{\sin^6 x}\cdot\tan x - \int\frac{(-6)}{\sin^7 x}\cdot\cos x\tan x\,dx$$

$$= \frac{\tan x}{\sin^6 x} + I_2 \Rightarrow I_1 - I_2 = \frac{\tan x}{\sin^6 x} + C$$

Thus, $\quad I = \dfrac{\tan x}{\sin^6 x} + C$

Hence, $\quad f(x) = \tan x$

29 Let $I = \int \tan^{-1}\sqrt{x}(1)\,dx$

$$= \tan^{-1}\sqrt{x}\int 1dx - \int\left[\int\frac{d}{dx}(\tan^{-1}\sqrt{x})\int(1)\,dx\right]dx$$

$$= \tan^{-1}\sqrt{x}\cdot x - \int\frac{1}{1+x}\times\frac{1}{2\sqrt{x}}.x\,dx$$

$$= x\tan^{-1}\sqrt{x} - \frac{1}{2}\int\frac{x}{(1+x)\sqrt{x}}dx$$

Put $x = t^2 \Rightarrow dx = 2t\cdot dt$

$$\therefore\ I = x\tan^{-1}\sqrt{x} - \frac{1}{2}\int\frac{t^2}{(1+t^2)\cdot t}2t\,dt$$

$$= x\tan^{-1}\sqrt{x} - \int\frac{t^2}{1+t^2}dt$$

$$= x\tan^{-1}\sqrt{x} - t + \tan^{-1}t + C$$

$$= x\tan^{-1}\sqrt{x} - \sqrt{x} + \tan^{-1}\sqrt{x} + C$$

$$= (x+1)\tan^{-1}\sqrt{x} - \sqrt{x} + C$$

30 $I_n = \displaystyle\int(\log x)^n dx = x(\log x)^n - n\int(\log x)^{n-1}\cdot\frac{1}{x}\cdot x\,dx$

$$\therefore\ I_n + nI_{n-1} = x(\log x)^n$$

31 Consider, $\int f^{-1}(x)dx = \int f^{-1}(x)\cdot 1dx$

$$= f^{-1}(x)\cdot x - \int\frac{d}{dx}(f^{-1}(x))\cdot x\,dx$$

Now, let $f^{-1}(x) = t$, then

$$\frac{d}{dx}(f^{-1}(x)) = \frac{dt}{dx}$$

$$\Rightarrow \quad \frac{d}{dx}(f^{-1}(x))\cdot dx = dt$$

$$\therefore \quad \int f^{-1}(x)dx = x\cdot f^{-1}(x) - \int f(t)dt$$

$[\because f^{-1}(x) = t \Rightarrow x = f(t)]$

$$= x\cdot f^{-1}(x) - g(t)$$

$$= x\cdot f^{-1}(x) - g(f^{-1}(x))$$

32 Let $I = \displaystyle\int e^x\left(\frac{x+3}{(x+4)^2}\right)dx$

$$= \int e^x\left(\frac{x+4-1}{(x+4)^2}\right)dx$$

$$= \int e^x\left(\frac{1}{x+4} - \frac{1}{(x+4)^2}\right)dx$$

$$= e^x\cdot\frac{1}{x+4} + C$$

$[\because \int e^x(f(x) + f'(x)dx = e^x\cdot f(x) + C]$

33 LHS $= \displaystyle\int\left[\frac{x^2+1}{x^2+1} - \frac{x}{x^2+1}\right]e^{\cot^{-1}x}dx$

$$= \int 1\cdot e^{\cot^{-1}x}dx - \int\frac{x}{x^2+1}e^{\cot^{-1}x}dx$$

$$= xe^{\cot^{-1}x} - \int x\cdot e^{\cot^{-1}x}\left(-\frac{1}{1+x^2}\right)dx$$

$$- \int\frac{x}{1+x^2}e^{\cot^{-1}x}dx + C = xe^{\cot^{-1}x} + C$$

$\therefore \quad A(x) = x$

34 We have, $\dfrac{d}{dx}\{g(x)\} = g(x)$

$$\Rightarrow \quad g'(x) = g(x)$$

$$\Rightarrow \quad \int\frac{g'(x)}{g(x)}dx = \int 1\,dx$$

$$\Rightarrow \quad \log_e\{g(x)\} = x + \log C_1$$

$$\Rightarrow \quad g(x) = C_1e^x$$

Now, $\quad g(0) = 1 \Rightarrow C_1 = 1$

$$\therefore \quad g(x) = e^x$$

$$\therefore \quad \int g(x)\left(\frac{2 - \sin 2x}{1 - \cos 2x}\right)dx$$

$$= \int e^x(\operatorname{cosec}^2 x - \cot x)\,dx$$

$$= -e^x\cot x + C$$

$$= -g(x)\cot x + C$$

35 Let $I = \displaystyle\int\frac{dx^3}{x^3(x^n+1)}dx = \int\frac{3x^2\,dx}{x^3(x^n+1)}dx$

$$= 3\int\frac{dx}{x(x^n+1)} = 3\int\frac{x^{n-1}\,dx}{x^n(x^n+1)}$$

On putting $x^n = t$, we get

$$I = \frac{3}{n}\int\frac{dt}{t(t+1)} = \frac{3}{n}\int\left[\frac{1}{t} - \frac{1}{t+1}\right]dt$$

$$= \frac{3}{n}[\log t - \log(t+1)] + C$$

$$= \frac{3}{n}\log\left(\frac{t}{t+1}\right) + C$$

$$= \frac{3}{n}\log\left(\frac{x^n}{x^n+1}\right) + C$$

SESSION 2

1 We have, $\int f(x)\,dx = f(x)$

$$\Rightarrow \quad \frac{d}{dx}\{f(x)\} = f(x)$$

$$\Rightarrow \quad \frac{1}{f(x)}d[f(x)] = dx$$

$$\Rightarrow \quad \log\{f(x)\} = x + \log C$$

$$\Rightarrow \quad f(x) = Ce^x$$

$$\Rightarrow \quad \{f(x)\}^2 = C^2e^{2x}$$

$$\therefore \quad \int\{f(x)\}^2\,dx = \int C^2e^{2x}dx$$

$$= \frac{C^2e^{2x}}{2} = \frac{1}{2}\{f(x)\}^2$$

2 We have,

$$f(x)=\begin{vmatrix}0 & x^2-\sin x & \cos x-2\\ \sin x-x^2 & 0 & 1-2x\\ 2-\cos x & 2x-1 & 0\end{vmatrix}$$

$$\Rightarrow f(x)=\begin{vmatrix}0 & \sin x-x^2 & 2-\cos x\\ x^2-\sin x & 0 & 2x-1\\ \cos x-2 & 1-2x & 0\end{vmatrix}$$

[interchanging rows and columns]

$$\Rightarrow f(x)=(-1)^3\begin{vmatrix}0 & x^2-\sin x & \cos x-2\\ \sin x-x^2 & 0 & 1-2x\\ 2-\cos x & 2x-1 & 0\end{vmatrix}$$

[taking (−1) common from each column]

$\Rightarrow \quad f(x)=-f(x) \Rightarrow f(x)=0$

$\therefore \quad \int f(x)\,dx=\int 0\,dx=C$

3 Let $I=\int e^{2ax}\frac{1-\cos 2ax}{1+\sin 2ax}dx$

$\Rightarrow I=\frac{1}{a}\int e^{2t}\frac{1-\cos 2t}{1+\sin 2t}dt$, [where, $ax=t$]

$$\Rightarrow I=\frac{1}{a}\int e^{2t}\frac{1-2\sin\left(\frac{\pi}{4}+t\right)\cdot\cos\left(\frac{\pi}{4}+t\right)}{2\sin^2\left(\frac{\pi}{4}+t\right)}dt$$

$$\Rightarrow I=\frac{1}{a}\int e^{2t}\left\{\frac{1}{2}\operatorname{cosec}^2\left(\frac{\pi}{4}+t\right)-\cot\left(\frac{\pi}{4}+t\right)\right\}dt$$

$$\Rightarrow I=\frac{1}{2a}\int e^{2t}\operatorname{cosec}^2\left(\frac{\pi}{4}+t\right)dt-\frac{1}{a}\int e^{2t}\cot\left(\frac{\pi}{4}+t\right)dt$$

$$\Rightarrow I=-\frac{1}{2a}e^{2t}\cot\left(\frac{\pi}{4}+t\right)+\frac{1}{a}\int e^{2t}\cot\left(\frac{\pi}{4}+t\right)dt-\frac{1}{a}\int e^{2t}\cot\left(\frac{\pi}{4}+t\right)dt+C$$

$\Rightarrow \quad I=-\frac{1}{2a}e^{2t}\cot\left(\frac{\pi}{4}+t\right)+C$

$\therefore \quad I=-\frac{1}{2a}e^{2ax}\cot\left(\frac{\pi}{4}+ax\right)+C$

4 We have,

$$\int x\sqrt{\frac{2\sin(x^2+1)-\sin 2(x^2+1)}{2\sin(x^2+1)+\sin 2(x^2+1)}}dx$$

$$=\int x\sqrt{\frac{2\sin(x^2+1)-2\sin(x^2+1)\cdot\cos(x^2+1)}{2\sin(x^2+1)+2\sin(x^2+1)\cdot\cos(x^2+1)}}dx$$

$$=\int x\sqrt{\frac{1-\cos(x^2+1)}{1+\cos(x^2+1)}}dx$$

$$=\int x\tan\left(\frac{x^2+1}{2}\right)dx$$

$$\therefore \int \tan\left(\frac{x^2+1}{2}\right)d\left(\frac{x^2+1}{2}\right)$$

$$=\log\left|\sec\left(\frac{x^2+1}{2}\right)\right|+C$$

5 We have,

$$I=\int\frac{\sin^2 x\cdot\cos^2 x}{(\sin^5 x+\cos^3 x\cdot\sin^2 x+\sin^3 x\cdot\cos^2 x+\cos^5 x)^2}dx$$

$$=\int\frac{\sin^2 x\cos^2 x}{\{\sin^3 x(\sin^2 x+\cos^2 x)+\cos^3 x(\sin^2 x+\cos^2 x)\}^2}dx$$

$$=\int\frac{\sin^2 x\cos^2 x}{(\sin^3 x+\cos^3 x)^2}dx$$

$$=\int\frac{\sin^2 x\cos^2 x}{\cos^6 x(1+\tan^3 x)^2}dx$$

$$=\int\frac{\tan^2 x\sec^2 x}{(1+\tan^3 x)^2}dx$$

Put $\tan^3 x=t$

$\Rightarrow \quad 3\tan^2 x\sec^2 x\,dx=dt$

$\therefore I=\frac{1}{3}\int\frac{dt}{(1+t)^2} \Rightarrow I=\frac{-1}{3(1+t)}+C$

$\Rightarrow \quad I=\frac{-1}{3(1+\tan^3 x)}+C$

6 Since,

$$\log\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)=\log\tan\left(\frac{\pi}{4}+\theta\right)$$

and $\int\sec\theta\,d\theta=\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$

$\Rightarrow \quad \int\sec 2\theta\,d\theta=\frac{1}{2}\log\tan\left(\frac{\pi}{4}+\theta\right)$

$2\sec 2\theta=\frac{d}{d\theta}\log\tan\left(\frac{\pi}{4}+\theta\right)$

$\therefore I=\frac{1}{2}\sin 2\theta\log\tan\left(\frac{\pi}{4}+\theta\right)-\int\tan 2\theta\,d\theta$

$=\frac{1}{2}\sin 2\theta\log\tan\left(\frac{\pi}{4}+\theta\right)-\frac{1}{2}\log\sec 2\theta+C$

7 Let $I=\int\frac{x^2}{(x\sin x+\cos x)^2}dx$

$$=\int\frac{x\cos x}{(x\sin x+\cos x)^2}\cdot\frac{x}{\cos x}dx$$

$\left[\because \frac{d}{dx}(x\sin x+\cos x)=x\cos x\right]$

$$\therefore \quad I=\frac{-1}{(x\sin x+\cos x)}\cdot\frac{x}{\cos x}+\int\frac{1}{(x\sin x+\cos x)}\cdot\frac{\cos x-x(-\sin x)}{\cos^2 x}dx$$

$$=\frac{-x}{(x\sin x+\cos x)\cos x}+\tan x+C$$

$$=\frac{-x+x\sin^2 x+\sin x\cdot\cos x}{(x\sin x+\cos x)\cdot\cos x}+C$$

$$=\frac{\sin x-x\cos x}{x\sin x+\cos x}+C$$

8 $f(x)=\int\frac{x^2\,dx}{(1+x^2)(1+\sqrt{1+x^2})}$

Put $x=\tan\theta \Rightarrow dx=\sec^2\theta\,d\theta$
$=(1+x^2)\,d\theta$

$\therefore \; f(x)=\int\frac{\tan^2\theta\sec^2\theta}{\sec^2\theta(1+\sec\theta)}d\theta$

$=\int\frac{1-\cos^2\theta}{\cos\theta\,(1+\cos\theta)}d\theta$

$=\int\sec\theta\,d\theta-\int d\theta$

$=\log(\sec\theta+\tan\theta)-\theta+C$

$=\log(x+\sqrt{1+x^2})-\tan^{-1}x+C$

$\Rightarrow \quad f(0)=\log(0+\sqrt{1+0})-\tan^{-1}(0)+C$

$\Rightarrow \quad C=0$

$\therefore \quad f(1)=\log(1+\sqrt{2})-\frac{\pi}{4}+0$

9 Let $I=\int\frac{dx}{(1-x)^2\sqrt[3]{\left(\frac{x+1}{1-x}\right)^2}}$

Put $\frac{1+x}{1-x}=t \Rightarrow \frac{2}{(1-x)^2}dx=dt$

$\therefore \quad I=\frac{1}{2}\int\frac{dt}{t^{2/3}}=\frac{3}{2}[t^{1/3}]+C$

$=\frac{3}{2}\left[\sqrt{\frac{1+x}{1-x}}+C\right]$

$\therefore \quad k=\frac{3}{2}$

10 $\int\frac{dx}{(\sin x+2)(\sin x-1)}$

$$=\frac{1}{3}\int\frac{dx}{(\sin x-1)}-\frac{1}{3}\int\frac{dx}{(\sin x+2)}$$

$$=\frac{1}{3}\int\frac{dx}{\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}-1\right)}-\frac{1}{3}\int\frac{dx}{\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}+2\right)}$$

Put $\tan\frac{x}{2}=t$

$\Rightarrow \quad \frac{1}{2}\sec^2\frac{x}{2}dx=dt$

$\therefore \quad \frac{1}{3}\int\frac{2dt}{2t-1-t^2}-\frac{1}{3}\int\frac{2dt}{2t+2t^2+2}$

$$= -\frac{2}{3}\int\frac{dt}{(t-1)^2} - \frac{1}{3}\int\frac{dt}{\left(t+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{2}{3}\frac{1}{(t-1)} - \frac{1}{3}\frac{2}{\sqrt{3}}\tan^{-1}\frac{\left(t+\frac{1}{2}\right)}{\frac{\sqrt{3}}{2}} + C$$

$$= \frac{2}{3}\frac{1}{\left(\tan\frac{x}{2}-1\right)} - \frac{2}{3\sqrt{3}}\tan^{-1}\frac{\left(\tan\frac{x}{2}+\frac{1}{2}\right)}{\frac{\sqrt{3}}{2}} + C$$

$$= \frac{2}{3\left(\tan\frac{x}{2}-1\right)} - \frac{2}{3\sqrt{3}}\tan^{-1}\left[\frac{2\left(\tan\frac{x}{2}+\frac{1}{2}\right)}{\sqrt{3}}\right] + C$$

11 $\int\frac{x^2}{(2+3x^2)^{5/2}}dx$

On substituting $2+3x^2 = t^2x^2$

$\Rightarrow \quad x^2 = \frac{2}{(t^2-3)}$

$\therefore \quad dx = -\frac{2t}{x(t^2-3)^2}dt$

$$\therefore \int\frac{x^2}{(tx)^5}\cdot\left(\frac{-2t}{x(t^2-3)^2}\right)dt = -2\int\frac{dt}{4t^4}$$

$$= -\frac{1}{2}\int\frac{dt}{t^4} = \frac{1}{6t^3} + C = \frac{1}{6}\left(\frac{x^2}{2+3x^2}\right)^{3/2} + C$$

12 $\int\left(1+x-\frac{1}{x}\right)e^{x+\frac{1}{x}}dx$

$$= \int e^{x+\frac{1}{x}}dx + \int x\left(1-\frac{1}{x^2}\right)e^{x+\frac{1}{x}}dx$$

$$= \int e^{x+\frac{1}{x}}dx + x\,e^{x+\frac{1}{x}} - \int\frac{d}{dx}(x)e^{x+\frac{1}{x}}dx$$

$$= \int e^{x+\frac{1}{x}}dx + xe^{x+\frac{1}{x}} - \int e^{x+\frac{1}{x}}dx$$

$$\left[\because \int\left(1-\frac{1}{x^2}\right)e^{x+\frac{1}{x}}dx = e^{x+\frac{1}{x}}\right]$$

$$= \int e^{x+1/x}dx + xe^{x+1/x} - \int e^{x+1/x}dx$$

$$= xe^{x+\frac{1}{x}} + C$$

13 Let $I = \int(\sin(101x)\cdot\sin^{99}x)dx$

$$= \int\sin(100x+x)\sin^{99}x\,dx$$

$$= \int(\sin(100x)\cdot\cos x + \cos(100x)\cdot\sin x)\sin^{99}x\,dx$$

$$= \int\sin 100x\cdot(\cos x\cdot\sin^{99}x)dx + \int\cos(100x)\cdot\sin^{100}x\,dx$$

$$= \left[\sin(100x)\cdot\frac{\sin^{100}x}{100} - \int\cos(100x)\cdot 100\cdot\frac{\sin^{100}x}{100}dx\right] + \int\cos(100x)\cdot\sin^{100}x\,dx$$

$$= \frac{\sin(100x)\cdot\sin^{100}x}{100} - \int\sin^{100}x\cdot\cos(100x)dx + \int\cos(100x)\cdot\sin^{100}x\,dx$$

$$= \frac{\sin(100x)\cdot\sin^{100}x}{100} + C$$

14 Let $I = \int\sqrt{\frac{(2018)^{2x}}{1-(2018)^{2x}}}\cdot(2018)^{\sin^{-1}(2018)^x}dx$

$$= \int\frac{(2018)^x}{\sqrt{1-(2018)^{2x}}}\cdot(2018)^{\sin^{-1}(2018)^x}$$

Put $\sin^{-1}(2018)^x = t$

$$\Rightarrow \frac{1}{\sqrt{1-(2018^x)^2}}\cdot(2018)^x l_n(2018)dx = dt$$

$$\therefore I = \frac{1}{\ln(2018)}\int(2018)^t dt$$

$$= \frac{1}{\ln(2018)}\cdot\frac{(2018)^t}{\ln 2018} + C$$

$$= \frac{(2018)^t}{\ln^2(2018)} + C$$

$$= (\log_{2018}e)^2\cdot(2018)^{\sin^{-1}(2018)^x} + C$$

15 Let $I = \int(\int e^x(\log x + \frac{2}{x} - \frac{1}{x^2})dx)dx$

$$= \int\left(\int e^x\left(\log x + \frac{1}{x} + \frac{1}{x} - \frac{1}{x^2}\right)dx\right)dx$$

$$= \int\left[\int e^x\left(\log x + \frac{1}{x}\right)dx + \int e^x\left(\frac{1}{x} - \frac{1}{x^2}\right)\right]dx$$

$$= \int\left(e^x\log x + e^x\frac{1}{x} + C_1\right)dx$$

$$[\because \int e^x(f(x) + f'(x))dx = e^x\cdot f(x) + C]$$

$$= \int e^x\left(\log x + \frac{1}{x}\right)dx + \int C_1\,dx$$

$$= e^x\log x + C_1x + C_2$$